Diagonalization of the length sensing matrix of a dual recycled laser interferometer
gravitational wave antenna

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Next generation gravitational wave (gw) antennas employ RSE (Resonant Sideband Extraction) interferometers with Fabry-Perot cavities in the arms as an optical configuration. In order to realize stable, robust control of the detector system, it is a key issue to extract appropriate control signals for longitudinal degrees of freedom of the complex coupled-cavity system. In this paper, a novel length sensing and control scheme is proposed for the tuned RSE interferometer that is both simple and efficient. The sensing matrix can be well diagonalized, owing to a simple allocation of two rf modulations and to a macroscopic displacement of cavity mirrors, which cause a detuning of the rf modulation sidebands.

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I. INTRODUCTION

Currently existing terrestrial interferometer gravitational wave (gw) antennas [1], which are categorized as first generation detectors, are currently going on line for searching gravitational waves. First detection and direct observation of gravitational waves are a primary goal and would be a landmark achievement in physics, however, the next ambition is opening a window to gravitational wave astronomy, a brand-new field of view to the universe. Differing from an electro-magnetic wave, the gw has only faint interaction with matter, so extremely high sensitivities, reaching or overcoming the standard quantum limit (SQL), are required for the next generation detectors. Planned detectors such as Advanced-LIGO (AdLIGO) [2] or the Japanese LCGT [3] can be considered second generation gw detectors, whose sensitivity is reaching to or will slightly surpass the SQL, by using various techniques, for example, an advanced optical configuration, advanced mirror materials, a different beam profile and cryogenics.

AdLIGO and LCGT will employ an optical configuration called “resonant sideband extraction” (RSE) [4], which is realized with an additional signal-recycling mirror added to the power-recycled Fabry-Perot Michelson interferometer (FPMI). It is placed between beamsplitter and dark port, as is shown in Fig. 1. The laser light resonates inside the two arm cavities and the power-recycling cavity, in a way identical to first generation gw detectors employing power-recycled FPMI.

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FIG. 1: The schematic of the RSE interferometer optical configuration. The signal extraction mirror is placed between the beamsplitter and the dark port to enhance the detector response, which makes the whole interferometer system more complex. The length degrees of freedom and the control signal extraction ports are shown in addition to the optical arrangement.

If the laser light is anti-resonant inside the signal-recycling cavity (without arm cavities), this is called “signal-recycling”, which is one of the operation modes of the dual-recycled FPMI. In contrast, if the carrier is resonant inside the signal-recycling cavity, as in the case...
of LCGT, this mode is called “resonant sideband extraction” (tuned RSE), and if the carrier is kept slightly off-resonant from tuned RSE (or signal-recycled FPMI) by microscopic displacement of the signal-recycling mirror position, it is called “detuned RSE (or detuned signal-recycling)” as in the case of AdLIGO.

Only when all of the test masses and other interferometer optics are kept at appropriate positions to make the laser light resonant and interfere properly inside the interferometer, the interferometer works as a gw antenna, producing linear signal proportional to the gw strain. Therefore, precise position control of optics is indispensable for keeping the interferometer operational. Therefore, one of the most important roles of the signal sensing scheme is to provide appropriate error signals for the degrees of freedom to be controlled. When the system is multidegree-of-freedom and needs multi-loop control, the signal mixture on each other could cause serious problems [5]. Therefore the primary goal of this study is to extract a diagonal sensing matrix, providing appropriate error signals independently without a signal mixture of other degrees of freedom.

In this paper, the emphasis is placed on the description of a sensing scheme that gives a diagonal sensing matrix, and an optimization of the optical configuration parameters of RSE using both analytical and numerical simulations. The numerical simulations were performed using Finesse simulation software [6]. This scheme was originally developed on the assumption of implementation for tuned RSE, however, it could also be applicable for detuned RSE.

II. REVIEW OF THE LENGTH SENSING AND CONTROL SCHEME

The baseline of the signal sensing method for the RSE interferometer is to use the Pound-Drever-Hall (PDH) [7] technique and its extended scheme: using different wavelengths of light which behave differently for each optical system. Therefore, the main issue of designing a signal sensing scheme is to determine the modulation and demodulation scheme, the arrangement of modulation sidebands. In this sense, the sensing scheme is associated one-to-one with the arrangement of the RF modulation(s); the number of the modulations, the resonance conditions of the modulations inside cavities and the Schnupp asymmetry factor.

For the first generation interferometers, an extended PDH technique is commonly used as a signal sensing method. The laser light is phase modulated at a rf frequency before illuminating the interferometer, then the light from the appropriate signal extraction ports is photo-detected and demodulated with a rf local signal to extract the sensing signals. Efficient signal extraction is achieved with intentional macroscopic asymmetry for the Michelson interferometer arm lengths, which is called pre-modulation or Schnupp modulation [8] technique for a gw interferometer application. The total number of longitudinal degrees of freedom to be controlled for the power-recycled Fabry-Perot Michelson interferometer (PRFPMI) configuration is four: common and differential length of the arm cavities, power-recycling cavity (PRC) length and Michelson differential length. In order to extract the four sensing signals, a single rf phase modulation and three signal extraction ports (bright, dark and pickoff ports) are dedicated.

On the other hand, for the RSE configuration, with the additional signal recycling mirror, there are five longitudinal degrees of freedom to be controlled, which are defined in Table I. The addition of the extra mirror makes the interferometer a more complex, multiply-coupled cavity system, which consists of arm cavities, power- and signal-recycling (extraction) cavities.

In connection with the additional degrees of freedom, a double-modulation and -demodulation scheme (DMD) was introduced to the gw laser interferometer. By introducing another set of modulation sidebands, there are increased combinations of beat signals between carrier and modulation sidebands. The conventional single modulation scheme (SMD) extracts the signals from the beating between carrier and the phase modulation sidebands, whereas DMD utilizes the beating between two modulation sidebands at different frequencies (see App. A).

Toward the firm realization of the advanced optical configuration, there were several table-top experiments of RSE systems to study the sensing and control method, and to demonstrate the feasibility of the dual-recycling with FP cavities; Caltech [9], Florida [10] and Australia [11]. These are all for a detuned operation of RSE, i.e. AdLIGO, and a prototype experiment on the 40 m interferometer at Caltech, and the 4 m interferometer at the National Astronomical Observatory of Japan are now in progress [12] [13].

For the most appropriate design of the signal sensing system, many factors should be carefully considered:

- **Diagonal sensing matrix**: It is ideal that five error signals for longitudinal degrees of freedom are extracted independently, without significant cross talk with other signals

- **Robust signal extraction**: The signal sensing matrix should be robust enough to any possible imperfections and asymmetry of the interferometer
• Less noise coupling: Some of the noises appear at the signal readout port in combination with the parameters which sensing design determined, so the noise coupling efficiency should be suppressed low enough

• Easy lock acquisition: The whole interferometer system should be rendered operational easily from an uncontrolled state. The dynamical nature of the five error signals are important during the lock acquisition process.

Suppose there is a system with \( i \) degrees of freedom that are to be controlled, and there is an identical number \( i \) of extracted signals. In general, each of these signals can have some sensitivity for all \( i \) degrees of freedom. However, they should be linearly independent from each other at least in principle, so that they could close stable feedback loops. Furthermore, it is well-known that the feedback signals should be as diagonal as possible to insure a robust control of the system.

There are two ways of diagonalization: optically diagonal state and electronically diagonal state. If the each of proper numbers of extracted signals is sensitive to only one of the degrees of freedom, it is called an optically diagonal state. Whereas, if the extracted signals are linearly independent, it is possible to reconstruct a diagonal state by manipulating the signals electronically (or digitally on the computer), which is referred to as an electronically diagonal state.

As far as control signals are concerned, both are thought to be almost equivalent; however, the optically diagonal state might have some potential advantages when the signal to noise ratio, dynamical nature of lock acquisition and robust control of the systems are taken into account. Therefore, among these points of view, the emphasis was especially placed on the optically diagonal sensing of the five error signals for longitudinal degrees of freedom, because it is true worth of the signal sensing. In the following sections, above items are discussed in connection with the newly proposed signal sensing scheme.

III. DIAGONALIZED SIGNAL SENSING

A. Allocation of modulations

\( L_+ \) and \( L_- \), the common and differential arm cavity length signals, are expected from the pre-modulation signal sensing scheme, beating the carrier and phase modulated sideband fields. The significant enhancement of the carrier field inside the two arm cavities results in huge phase sensitivities in extracting the signals for \( L_+ \) and \( L_- \). This means both \( L \)-signals can be extracted relatively independently, without much mixture of any other longitudinal signals, or in other words, the sensing matrix for them is almost diagonal. On the other hand, for the \( I_p \), \( I_s \) and \( L \)-signals of the central part of the RSE interferometer, it is known from preceding studies that extracting independent signals is not easy. Therefore, the main issue here is to spot the sensing scheme that enables a diagonal sensing matrix.

By analyzing the optical configuration of RSE closely, the complicated optical system turns out to be an intricately connected cavity system. Furthermore, when one concentrates on the central part of the interferometer, consisting of input test mass, recycling mirrors and beam splitter, the dual-recycled Michelson interferometer can be regarded as a coupled cavity (Fig. 2). Thus the issue is now reduced to the signal extraction of the coupled cavity.

One of the important features of the PDH scheme is to make use of different responses (or, equivalently, different field enhancements) of the optical fields inside optical systems. Therefore, a different arrangement of the two sets of sidebands for the coupled cavity is essential. The most simple, straightforward and natural allocation would be that one of sidebands circulates both inside PRC and SEC, whereas the other sideband resonates only inside the PRC, as is shown in Fig. 2. The \( L_p \) signal could be extracted due to the above difference between the two sidebands, in contrast, the \( L_s \) signal could be obtain making use of the difference of field enhancements inside the PRC.

The concepts of the suggested signal extraction scheme can be summarized in the following:

- PM (phase modulation) sidebands are completely transmitted by the MI and resonate inside the PRC+SEC
- AM (amplitude modulation) sidebands are completely reflected by the MI, and resonate only inside the PRC.

The PM should circulate inside both PRC and SEC to reach to the dark port so that it can be used as a local field against the carrier field for the \( L_- \) signal extraction using a conventional PDH technique; whereas the AM can be a local field against the PM field on double demodulation scheme. The second modulation must be AM in case of tuned RSE so that the beating between two modulations can produce error signals. However, it can be PM again in the case of detuned RSE, because the detuning of the carrier inside the SEC can make the phase modulation sidebands unbalanced resonance inside cavities, which could produce beat signals. The point of this modulation scheme is that the arrangement of the modulation sidebands is very simple, using complete transmission and reflection at the Michelson interferometer, which enables clean signal extractions especially for the \( L \)-signals.

In order to realize the sidebands allocation as shown in Fig. 2, the optical property of the coupling mirror (Michelson interferometer for the case of RSE) of the coupled cavity system could be a key factor. The major difference between a simple coupled cavity and the central part of RSE is that the coupling mirror could have
variable reflectivities. The reflectivities of the Michelson interferometer for the RSE for modulation sidebands are functions of the \( j \)-th modulation frequency \( \omega_j \) and the Schnupp asymmetry \( \delta_{\text{sch}} \), and they are defined as \( r_{\text{MI}j} = \cos \alpha_j \) and \( t_{\text{MI}j} = i \sin \alpha_j \) with an asymmetry factor \( \alpha_j = \delta_{\text{sch}} \omega_j / c \). Therefore, in general, two sets of sideband fields with different modulation frequency may have different reflectivities and transmissivities for the Michelson interferometer. For PM and AM, satisfying the above conditions, the following constraints are imposed:

- **PM**: \( r_{\text{MI}1} = 0 \) with \( \delta_{\text{sch}} \omega_{m1} / c = \pi / 2 + n \pi \)
- **AM**: \( r_{\text{MI}2} = 1 \) with \( \delta_{\text{sch}} \omega_{m2} / c = n' \pi \)

where \( n \) and \( n' \) are integers. The control signal extraction ports are assumed to be at bright, pickoff and dark port (transmitted light from SEM), which is the same as for first generation gw antennas. Using a double-modulation and -demodulation procedure, as is described in Appendix A, the analytic expressions for signal sensitivities on longitudinal degrees of freedoms are given as follows. At the bright port, the signal sensitivities for \( l_p, l_- \) and \( l_s \) are given as

\[
\frac{\partial V_{\text{pd}}}{\partial l_p} \propto S \left( g_1^2 r_{\text{a}1}^2 r_3 r_{b2} + r_{b1} g_2^2 r_{\text{a}2} \right) \cos \delta_1 \cos \delta_2
\]

\[
\frac{\partial V_{\text{pd}}}{\partial l_s} \propto S \left( g_1^2 r_{\text{a}1}^2 r_3 r_{b2} \right) \cos \delta_1 \cos \delta_2
\]

\[
\frac{\partial V_{\text{pd}}}{\partial l_-} \propto 4 g_1^2 r_{\text{a}1} r_{b2} (1 - r_{\text{a}2}^2) \sin \delta_1 \cos \delta_2
\]

respectively. The signals at pick-off port contains similar information as follows,

\[
\frac{\partial V_{\text{dd}}}{\partial l_p} \propto S \left( r_p g_1 r_2 g_1 r_3 r_{\text{a}1}^2 + g_2 r_{\text{a}2} \right) \cos \delta_1 \cos \delta_2
\]

\[
\frac{\partial V_{\text{dd}}}{\partial l_s} \propto S \left( r_p g_1 r_2 r_{\text{a}1}^2 \cos \delta_1 \cos \delta_2
\]

\[
\frac{\partial V_{\text{dd}}}{\partial l_-} \propto 4 r_p g_1 r_2 r_{\text{a}1} (1 + r_{\text{a}2}^2) \sin \delta_1 \cos \delta_2
\]

with slight differences that can be used to determine the polarity of the signal. In contrast, for the signals at the dark port, the signals are given as

\[
\frac{\partial V_{\text{dd}}}{\partial l_p} = 0
\]

\[
\frac{\partial V_{\text{dd}}}{\partial l_s} = 0
\]

\[
\frac{\partial V_{\text{dd}}}{\partial l_-} \propto 4 g_1 r_2 r_{\text{a}1} r_2 g_2 r_{\text{a}2} \sin \delta_1 \cos \delta_2
\]

Here \( r_x \) and \( t_x \) are the amplitude reflectivity and transmissivity of optics (\( x=p, s \) for power recycling and signal extraction mirror, respectively), whereas \( r_{xy} \) are an optical systems, for field “y” at port “x”. The reflectivity of the arm cavities for the PM sidebands (sideband 1), for example, is given as \( r_{a1} = (1 - r_{b1} r_{a2}) / (1 + r_{b1} r_{a2}) \), where exact anti-resonance is supposed inside the arm cavities. On the other hand, \( g_{xy} \) is a field enhancement factor for field “y” inside cavity “x”. The field enhancement factor is a measure of how much the field is amplified inside the cavity, which usually gives a measure of the phase sensitivity to the signal, so it is a very important factor. For example, \( g_{a1} = t_p / (1 - r_{b1} r_{a2}) \) is for the PM sidebands inside the power recycling cavity (and also inside the SEC, because they are exactly identical in this case); \( g_{a2} = t_p / (1 + r_{b1} r_{a2}) \) is for the PM sidebands inside the power recycling cavity. The demodulation phases are denoted with \( \delta_i \), \( i = 1, 2 \) for PM (modulation 1) and AM (modulation 2).

When the carrier and the modulation sidebands are on exact resonance or anti-resonance inside the arm cavities, PRC and SEC, all parameters appearing in the above expressions become real quantities, so the signal sensitivities of demodulated signals depend only on the demodulation phase \( \delta_i \). For example for \( l_p \) and \( l_s \), both error signals show the same dependences on the \( \delta_i \), which means that the two signals are completely “linearly dependent”, in terms of demodulation phases, though the

\[ \text{FIG. 2: Resonant conditions for optical fields, carrier and modulation sidebands. The carrier field resonates inside both arm cavities and the power-recycling cavity, which gives enhanced responses to the gravitational wave signals. The phase modulation is designed to circulate inside the power-recycling and the signal-extraction cavity, whereas the amplitude modulation circulates only inside the power-recycling cavity. The central part of the RSE, dual recycled Michelson interferometer can be viewed as a coupled cavity, as is shown in the lower figure. The coupling mirror, which connects the two cavities, is a Michelson interferometer instead of plane mirror.} \]
signal sensitivities are slightly different due to the finesse difference between PRC and SEC. Therefore, in this situation, it is quite difficult to extract two independent error signals for the PRC and SEC lengths.

B. "Delocation"

The idea to resolve this question is to make use of an off-resonant condition of the light field(s). The simple way to realize this condition is to make the coupling cavity off-resonant for rf sidebands by a macroscopic detuning, as is shown in Fig. 3. The resonance condition of the carrier in the two cavities are maintained whereas the rf sidebands are slightly detuned inside the power-recycling cavity by changing the position of PRM [14].

There are two effects: one is a slight off-resonance condition for AM appearing on $g_{2u2}$, $r_{2u2}$ and $\Delta_2$, the other is macroscopically extended PRC length for PM, typically appearing on $\Delta_1$. Now $g_{2u2} = t_p/(1 + r_p r_{2u2} e^{-i\Delta_2})$ and $r_{2u2} = (r_p + r_{2u2} e^{-i\Delta_2})/(1 + r_p r_{2u2} e^{-i\Delta_2})$, field enhancement factor and reflectivity at the bright port for AM sidebands respectively, have complex values instead of real values. The "delocation" effect is included in the delocation phase $\Delta_1 = \omega_0 \Delta_1 / c$ and $\Delta_2 = \omega_0 \Delta_2 / c$, where $\Delta_1$ is a macroscopic "delocation". The signals at pick-off port are again similar to that at bright port as follows:

$$\frac{\partial V_{pd}}{\partial l_p} \propto 8 \Im \left[ g_{2u2} \left( g_{1u} r_{s1} r_{a1} + g_{2u2} r_{2u2} e^{-i\Delta_2} \right) e^{-i\delta_1} \right] \cos \delta_1$$

$$\frac{\partial V_{pd}}{\partial l_s} \propto 8 \Im \left[ g_{1u} r_{s1} r_{a1} e^{-i\Delta_1} \right] \cos \delta_1$$

$$\frac{\partial V_{pd}}{\partial l_\perp} \propto -4 t_p g_{2u2} r_{2u2} e^{-i\Delta_2} \left[ (e^{i\Delta_1} - r_{s1} r_{2u2} e^{-i\Delta_2}) e^{-i\delta_1} \right] R \left[ g_{2u2} e^{-i\delta_1} \right]$$

As for the signals at dark port,

$$\frac{\partial V_{dd}}{\partial l_p} = 0$$

$$\frac{\partial V_{dd}}{\partial l_s} = 0$$

$$\frac{\partial V_{dd}}{\partial l_\perp} \propto 4 t_p g_{2u2} r_{2u2} e^{-i\Delta_2} \sin \delta_1.$$
TABLE II: The numerical results of the length sensing matrix. Five signals are extracted from each port, using particular demodulation scheme with some demodulation phase(s) to provide appropriate error signals. The matrix elements are normalized so that the diagonal elements become unity. The numerical result using FINESSE software are also listed between parentheses for comparison. Design values for LCGT were used for this calculation as optical parameters; \( r_1 = r_3 = 0.996000, r_2 = r_4 = 0.999950, r_p = 0.80 \) and \( r_s = 0.77 \). Through these calculations, the interferometer was assumed to be lossless for simplicity. As for matrix elements, “0” means analytically zero, whereas “0.00” means numerically zero.

<table>
<thead>
<tr>
<th>Port</th>
<th>Demodulation</th>
<th>( \delta_1 )</th>
<th>( \delta_2 )</th>
<th>( \partial / \partial \Phi_c )</th>
<th>( \partial / \partial \Phi_p )</th>
<th>( \partial / \partial \phi_{o_1} )</th>
<th>( \partial / \partial \phi_{o_2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V_B )</td>
<td>SD(CR&amp;PM)</td>
<td>-1.06</td>
<td>-</td>
<td>(1)</td>
<td>(0.00)</td>
<td>(0.00)</td>
<td>(2.56 \times 10^{-3})</td>
</tr>
<tr>
<td>( V_B )</td>
<td>SD(CR&amp;PM)</td>
<td>90</td>
<td>-</td>
<td>0</td>
<td>1</td>
<td>(0.00)</td>
<td>(0.00)</td>
</tr>
<tr>
<td>( V_P )</td>
<td>DD(PM&amp;AM)</td>
<td>-3.32</td>
<td>69.2</td>
<td>1.00 \times 10^{-3}</td>
<td>0</td>
<td>(0.00)</td>
<td>(1.89 \times 10^{-6})</td>
</tr>
<tr>
<td>( V_D )</td>
<td>DD(PM&amp;AM)</td>
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<td>(-1.00 \times 10^{-3})</td>
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<td>(1)</td>
<td>(1.89 \times 10^{-6})</td>
</tr>
<tr>
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<td>DD(PM&amp;AM)</td>
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<td>27.99</td>
<td>1.00 \times 10^{-3}</td>
<td>0</td>
<td>(0.00)</td>
<td>(1.28 \times 10^{-7})</td>
</tr>
</tbody>
</table>

sensing matrix, which is another significant merit of this sensing scheme.

The numerical result of the sensing matrix is shown in Table II. The signal sensitivity given by analytical expressions are listed together with the numerical results using the FINESSE simulation software. Analytical expressions for \( L \)-signals and \( l \)-signals with single demodulation are given in Appendix C. The off-diagonal values show quite a good agreement between the two methods. Slight differences in some of the off-diagonals are due to the different resonant conditions of two sets of sidebands for the arm cavities.

IV. CONCLUSION

A novel longitudinal sensing scheme for tuned RSE was newly proposed. In order to extract the error signals for the central part of the RSE interferometer, phase and amplitude modulations are suggested for the double modulation scheme. Based on the concept of the PDH scheme, very simple allocations of the two sets of modulation sidebands are implemented, which enables an almost diagonal sensing matrix, and is improved further by the “delocation” technique.

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APPENDIX A: DOUBLE-MODULATION AND DEMODULATION

In general, double modulation could be an arbitrary combination of phase and amplitude modulations, however, in this paper phase and amplitude modulations are assumed for double modulation.

Incoming laser light (carrier) is modulated using electro-optic modulators (EOMs) to produce modulation sidebands. To avoid the generation of sub-sidebands (sideband of sideband), parallel modulation instead of series modulation should be used. As a result, input fields will take the form of

\[
E_{inc} = E_0 e^{i\phi_{ct}}
\]

\[
\rightarrow E_0 (1 + \Gamma_2 \cos(\omega_2 t)) e^{i(\Omega_0 t + \Gamma_1 \cos(\omega_1 t))}
\]

\[
\approx E_0 e^{i\phi_{ct}} \left\{ 1 + i \Gamma_1 \left( e^{i\omega_1 t} + e^{-i\omega_1 t} \right) + \frac{\Gamma_2}{2} \left( e^{i\omega_2 t} + e^{-i\omega_2 t} \right) \right\}
\]

\[
\equiv E_0 e^{i\phi_{ct}} + E_1 e^{i(\Omega_0 + \omega_1) t} + E_1 e^{i(\Omega_0 - \omega_1) t} + E_2 e^{i(\Omega_0 + \omega_2) t} + E_{2e} e^{i(\Omega_0 - \omega_2) t}
\]

where the modulation depths \( \Gamma_1 \) for both modulations were assumed to be small enough so that the above approximation holds. Then these five frequencies of electric fields will have different responses from an optical system (RSE interferometer, in this case) depending on the optical frequencies. So the output fields will have the form of

\[
E_{out} = -E_0 H_{xy} e^{i\phi_{ct}}
+ E_1 H_{x1e} e^{i(\Omega_0 + \omega_1) t} + E_1 H_{x1e} e^{i(\Omega_0 - \omega_1) t}
+ E_2 H_{x2e} e^{i(\Omega_0 + \omega_2) t} + E_{2e} H_{x2e} e^{i(\Omega_0 - \omega_2) t}
\]

where \( H_{xy} \) is a response of the field “y” (0 for carrier, 1U for upper and 1L for lower sideband) from incident point to the signal port “x”. 

\[ \Phi \]
The reflection and pick-off ports are dedicated as signal extraction ports, from which portions of the light fields are picked up, received with photo detector, and then demodulated by a mixer with a local rf signal to produce error signals. The demodulated signals, which are extracted from beat signals between carrier and the phase modulation sidebands are given in general form as

\[ V_x = 2\Re \left[ (E_1 H_{x1})^* E_2 H_{x2u} + (E_2 H_{x2})^* E_1 H_{x1u} \right] e^{-i\delta_1} e^{-i\delta_2} + ((E_1 H_{x1})^* E_2 H_{x2u} + (E_2 H_{x2})^* E_1 H_{x1u}) e^{i\delta_1} e^{-i\delta_2} \]

where \( x \) denotes signal port, and \( \delta_1 \) is a demodulation phase; \( \delta_1 = 0 \) corresponds to in-phase and \( \delta_1 = \pi/2 \) to quadrature phase demodulation.

\[ \frac{\partial V_x}{\partial \phi_x} \propto \Im \left[ \frac{\partial}{\partial \phi_z} (H_{x1u} H_{x2u} - H_{x2u} H_{x1u}) e^{-i\delta_1} e^{-i\delta_2} + \frac{\partial}{\partial \phi_z} (H_{x1u} H_{x2u} - H_{x2u} H_{x1u}) e^{i\delta_1} e^{-i\delta_2} \right] \]

where \( z \) is the phase degree of freedom associated with the cavity lengths. The ideal demodulation process is assumed here, so all detection and demodulation efficiencies are set to be unity.

**APPENDIX B: COMPLETE EXPRESSIONS OF THE FIELDS OF RSE INTERFEROMETER**

The optical-frequency-dependant transfer functions of the light fields from the input to all three signal ports are

\[ H_{by}(\Phi_1, \phi_1) = -r_p + \left\{ \frac{1}{2} \left( r_{a2}(\Phi_2) e^{-i\phi_2} + r_{a1}(\Phi_1) e^{-i\phi_1} \right) + \frac{1}{2} r_a \left( r_{a2}(\Phi_2) e^{-i\phi_2} r_{a1}(\Phi_1) e^{-i\phi_1} \right) \right\} \]

\[ H_{py}(\Phi_1, \phi_1) = \frac{t_p}{1 - r_p} \left\{ \frac{1}{2} \left( r_{a2}(\Phi_2) e^{-i\phi_2} + r_{a1}(\Phi_1) e^{-i\phi_1} \right) + \frac{1}{2} r_a \left( r_{a2}(\Phi_2) e^{-i\phi_2} r_{a1}(\Phi_1) e^{-i\phi_1} \right) \right\} \]

\[ H_{dy}(\Phi_1, \phi_1) = \frac{t_p t_s}{1 - r_p} \left\{ \frac{1}{2} \left( r_{a2}(\Phi_2) e^{-i\phi_2} + r_{a1}(\Phi_1) e^{-i\phi_1} \right) + \frac{1}{2} r_a \left( r_{a2}(\Phi_2) e^{-i\phi_2} r_{a1}(\Phi_1) e^{-i\phi_1} \right) \right\} \]

where \( \phi_i = 2L_i \Omega/c \) and \( \Phi_i = 2L_i \Omega/c \) are the round trip phase of the arm and recycling cavities. The definitions for \( L_i, t_i \) are given in Fig. 1. \( r_{a1} \) is the reflectivity of the arm cavities which is given as

\[ r_{a1}(\Phi_1) = \frac{-r_1 + r_2 e^{-i\phi_1}}{1 - r_1 r_2 e^{-i\phi_1}} \]

\[ r_{a2}(\Phi_2) = \frac{-r_3 + r_4 e^{-i\phi_2}}{1 - r_3 r_4 e^{-i\phi_2}} \]

\[ r_a = \left( \frac{1}{r_3} + \frac{2}{r_4} e^{-i\phi_2} \right) \]
APPENDIX C: L-SIGNALS AND l-SIGNALS WITH SINGLE DEMODULATION

Using these transfer functions and the demodulation procedure (Appendix A), analytic expressions for signal sensitivities are calculated. The bright and dark port signals for the L-signals using single demodulations are given as

\[ \frac{\partial V_{\text{bs}}}{\partial L_+} \propto 2r_2(g_{0b}^2g_{0a}^2r_0r_1 + 2r_0g_{0b}^2g_{1a}^2r_0r_1) \cos \delta_1 \]
\[ \frac{\partial V_{\text{bs}}}{\partial L_-} \propto 2r_0g_{0b}^2g_{1a}^2r_2 \delta \left[(e^{i\Delta_1} - r_{1a}^2e^{-i\Delta_1})e^{-i\delta_1}\right] \]
\[ \frac{\partial V_{\text{bs}}}{\partial L_p} \propto 4(g_{0a}^2r_0a_0r_0 + r_0g_{0b}^2r_1r_0^2) \cos \delta_1 \]
\[ \frac{\partial V_{\text{bs}}}{\partial a_s} \propto 2r_0g_{0b}^2r_1a_0\delta \left[(e^{i\Delta_1} - r_{1a}^2e^{-i\Delta_1})e^{-i\delta_1}\right] \]

\[ \frac{\partial V_{\text{bs}}}{\partial L_+} = 0 \]
\[ \frac{\partial V_{\text{bs}}}{\partial L_-} \propto -4g_{0b}g_{0a}^2g_{0a}^2t_3r_0r_1 \sin \delta_1 \]
\[ \frac{\partial V_{\text{bs}}}{\partial L_p} = 0 \]
\[ \frac{\partial V_{\text{bs}}}{\partial a_s} = 0 \]
\[ \frac{\partial V_{\text{bs}}}{\partial a_s} \propto 2g_{0b}g_{0a}g_{0a}t_3r_0r_1 \sin \delta_1 \]

The sensitivities of the double demodulated signals for L-signals are given as

\[ \frac{\partial V_{\text{dl}}}{\partial L_+} \propto 4r_2 \Re \left[ (2g_{0b}^2g_{0a}^2r_1a_1r_0^2 + r_{b1}g_{2a}^2e^{-i\Delta_2})e^{-i\delta_1}\right] \cos \delta_1 \]
\[ \frac{\partial V_{\text{dl}}}{\partial L_-} \propto -4\Im \left[ g_{1a}^2g_{0a}^2r_2(e^{-i\Delta_1} - r_{1a}^2e^{i\Delta_1})e^{-i\delta_1}\right] \Re \left[ r_{b2a}e^{-i\delta_2}\right] \]

\[ \frac{\partial V_{\text{dl}}}{\partial L_+} \propto 4r_1^2 \Re \left[ g_{2a}(2g_{r1}g_{a}^2r_1a_1r_0r_0 + g_{2a}g_{a}^2e^{-i\Delta_2})e^{-i\Delta_1}\right] \cos \delta_1 \]
\[ \frac{\partial V_{\text{dl}}}{\partial L_-} \propto -4r_1^2 \Im \left[ g_{1a}^2g_{0a}^2r_2 \delta \left[(e^{-i\Delta_1} - r_{1a}^2e^{i\Delta_1})e^{-i\delta_1}\right] \Re \left[ r_{b2a}e^{-i\delta_2}\right] \]

\[ \frac{\partial V_{\text{dl}}}{\partial a_s} = 0 \]
\[ \frac{\partial V_{\text{dl}}}{\partial a_s} \propto -8r_1^2t_3r_0a_0g_{0a}^2 \Im \left[ g_{2a}g_{2a}e^{-i\delta_2}\right] \sin \delta_1 \]